3.11 Sum and Difference Identities

Cosine of a Sum or Difference:

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$(\textit{Note the sign switch in either case.})$$

Ex.

Find the exact value of $\cos 15^\circ$ without using a calculator.

Solution: Since 15° is not an angle on the unit circle we will use a sum or difference of two angles on the unit circle that will give us 15° and then use the formulas above to find the exact value.

$45^\circ - 30^\circ = 15^\circ$ so, we will use the difference formula.

$$\cos 15^\circ = \cos(45^\circ - 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)$$

Sine of a Sum or Difference

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$(\textit{Note the sign does not switch in either case.})$$

Ex.

Write each of the following expressions as the sine or cosine of an angle.

a) $\sin 22°\cos 13° + \cos 22°\sin 13°$

The key here is to recognize which of the formulas apply. Looking at the sine of a sum formula we can see that this is the formula which applies.

Therefore,

$\sin 22°\cos 13° + \cos 22°\sin 13°$

$= \sin(22° + 13°) = \sin 35°$

b) $\cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$ (cosine of diff. form.)

$= \cos(\frac{\pi}{3} - \frac{\pi}{4}) = \cos\frac{\pi}{12}$

Tangent of a Difference or Sum

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Which is derived from the sine and cosine formulas:

$$\tan(u \pm v) = \frac{\sin(u \pm v)}{\cos(u \pm v)} \frac{\sin u \cos v \pm \cos u \sin v}{\cos u \cos v \mp \sin u \sin v}$$

Ex.

Prove the reduction formula:

$$\tan\left(\theta - \frac{3\pi}{2}\right) = -\cot\theta$$

We need to use the formula with sine and cosine. Why?

So, $\tan\left(\theta - \frac{3\pi}{2}\right) = \dfrac{\sin\left(\theta - \frac{3\pi}{2}\right)}{\cos\left(\theta - \frac{3\pi}{2}\right)}$

$$= \frac{\sin\theta\cos\frac{3\pi}{2} - \cos\theta\sin\frac{3\pi}{2}}{\cos\theta\cos\frac{3\pi}{2} + \sin\theta\sin\frac{3\pi}{2}}$$

$$= \frac{\sin\theta\cdot 0 - \cos\theta\cdot(-1)}{\cos\theta\cdot 0 + \sin\theta\cdot(-1)} = -\cot\theta$$